Algorithms for Computing Ambisonics Translation Filters

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Summary

In this report, we distill and reproduce the necessary algorithms for computing ambisonics translation filters. These algorithms are also described by Gumerov and Duraiswami [1, chapter 3] for complex-valued spherical harmonics and by Zotter [2, chapter III] for both complex- and real-valued spherical harmonics. The translation operation consists of three steps: 1) rotating the coordinate system to align the z-axis with the desired translation direction, 2) translating along the new z-axis, and 3) rotating the coordinate system back to its original orientation. In Section 1, we specify the mathematical definitions and conventions followed in this report and, in Section 2, we present a mathematical formulation of the filters for translating ambisonics signals. Next, in Section 3, we present recurrence coefficients which will be used both in Section 4 to derive the rotation matrices needed to align the z-axis and in Section 5 to derive the coefficient matrix used to translate along the z-axis. We then show in Section 6 how to combine these individual matrices in order to compute the coefficients matrix for an arbitrary translation.

1 Definitions and conventions

As is common in higher-order ambisonics, we adopt Cartesian and spherical coordinate systems in which, for a listener positioned at the origin, the $+x$-axis points forward, the $+y$-axis points to the left, and the $+z$-axis points upward. Correspondingly, r is the (nonnegative) radial distance from the origin, $\theta \in [-\pi/2, \pi/2]$ is the elevation angle above the horizontal (x-y) plane, and $\phi \in [0, 2\pi)$ is the azimuthal angle around the vertical (z) axis, with $(\theta, \phi) = (0, 0)$ corresponding to the $+x$ direction and $(0, \pi/2)$ to the $+y$ direction. For a position vector $\vec{r} = (x, y, z)$, we denote the corresponding unit vector by $\hat{r} \equiv \vec{r}/r$.

Here, we use real-valued orthonormal (N3D) spherical harmonics, as given by Zotter [2, section 2.2], and we adopt the ambisonics channel number (ACN) convention [3] such that, for a spherical harmonic function of degree $l \in [0, \infty)$ and order $m \in [-l, l]$, the ACN index n is given by $n = l(l + 1) + m$ and the spherical harmonic function is denoted by Y_n.

In the free field (i.e., in a region free of sources and scattering bodies), the *acoustic potential field*, ψ (defined as the Fourier transform of the acoustic pressure field), satisfies the homogeneous Helmholtz equation, and can therefore be expressed as an infinite sum of regular (i.e., not singular) basis solutions. In ambisonics, these basis solutions are given by $j_l(kr)Y_n(\hat{r})$, where j_l is the spherical Bessel function of order l, and the sum, also known as a spherical Fourier-Bessel series expansion, is given by [1, chapter 2]

$$\psi(k, \vec{r}) = \sum_{n=0}^{\infty} 4\pi(-i)^l A_n(k) j_l(kr) Y_n(\hat{r}), \tag{1}$$

where A_n are the corresponding (frequency-dependent) expansion coefficients and we have, without loss of generality, factored out $(-i)^l$ to ensure conjugate-symmetry in each A_n, making each ambisonics signal (i.e., the inverse Fourier transform of A_n) real-valued for a real pressure field. In practice, this expansion is truncated to a finite order L (i.e., $l \in [0, L]$), yielding $N = (L+1)^2$ terms.

2 Ambisonics translation coefficients

Let $R_n(k, \vec{r})$ denote the spherical Fourier-Bessel basis functions defined in Eq. (1) and given by

$$R_n(k, \vec{r}) = 4\pi(-i)^l j_l(kr) Y_n(\hat{r}). \tag{2}$$

It can be shown that these basis functions can be translated along the vector $\vec{r}_0$ by [1, Eq. (3.2.1)]

$$R_n(k, \vec{r} + \vec{r}_0) = \sum_{n'=0}^{\infty} T_{n,n'}(k, \vec{r}_0) R_{n'}(k, \vec{r}), \tag{3}$$

where $T_{n,n'}$ are the so-called *translation coefficients*. Integral forms of these translation coefficients as well as fast recurrence relations for computing them are given by Gumerov and Duraiswami [1, section 3.2] and Zotter [2, chapter 3].

Now let us consider two sets of spherical Fourier-Bessel expansion coefficients for the same sound field: B_n, which describe the sound field for an expansion about the origin, and $A_{n'}$, which do the same for an expansion about $\vec{r}_0$. By Eq. (1), the acoustic potential field is given by

$$\psi(k, \vec{r} + \vec{r}_0) = \sum_{n'=0}^{\infty} A_{n'}(k) R_{n'}(k, \vec{r}) = \sum_{n=0}^{\infty} B_n(k) R_n(k, \vec{r} + \vec{r}_0). \tag{4}$$

Substituting Eq. (3) into the above equation yields

$$\sum_{n'=0}^{\infty} A_{n'}(k) R_{n'}(k, \vec{r}) = \sum_{n=0}^{\infty} B_n(k) \left[\sum_{n'=0}^{\infty} T_{n,n'}(k, \vec{r}_0) R_{n'}(k, \vec{r}) \right]. \tag{5}$$

Rearranging and interchanging the order of the summations[1] reveals

$$\sum_{n'=0}^{\infty} A_{n'}(k) R_{n'}(k, \vec{r}) = \sum_{n'=0}^{\infty} \left[\sum_{n=0}^{\infty} T_{n,n'}(k, \vec{r}_0) B_n(k) \right] R_{n'}(k, \vec{r}), \tag{6}$$

$$\implies A_{n'}(k) = \sum_{n=0}^{\infty} T_{n,n'}(k, \vec{r}_0) B_n(k). \tag{7}$$

[1] Note that this is possible when the summations converge absolutely and uniformly [1, section 3.1.1], which is likely the case since the expression necessarily converges to a finite ψ for real sound fields. In any case, since in practice these summations will be truncated at a finite order, swapping the order becomes trivial.

In practice, we have only a truncated series expansion of the sound field, with measured coefficients B_n up to order L, such that we compute

$$A_{n'}(k) = \sum_{n=0}^{N-1} T_{n,n'}(k, \vec{r}_0) B_n(k), \tag{8}$$

where $N = (L+1)^2$. Note that the translated expansion coefficients $A_{n'}$ can be computed to an arbitrary order L', with $N' = (L'+1)^2$ terms.

In matrix form, we can write Eq. (8) as

$$\mathbf{a}(k) = (\mathbf{T}(k, \vec{r}_0))^{\mathrm{T}} \cdot \mathbf{b}(k), \tag{9}$$

where $(\cdot)^{\mathrm{T}}$ denotes the transpose of the argument and, omitting dependencies,

$$\mathbf{a} = \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_{N'-1} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_{N-1} \end{bmatrix}, \tag{10}$$

and

$$\mathbf{T}_{(N\times N')} = \begin{bmatrix} T_{0,0} & T_{0,1} & \cdots & T_{0,N'-1} \\ T_{1,0} & T_{1,1} & \cdots & T_{1,N'-1} \\ \vdots & \vdots & \ddots & \vdots \\ T_{N-1,0} & T_{N-1,1} & \cdots & T_{N-1,N'-1} \end{bmatrix}. \tag{11}$$

In the following sections, we describe how to compute these translation coefficients for an arbitrary $\vec{r}_0$ via recurrence formulae.

3 Recurrence coefficients

The a and b recurrence coefficients are given, respectively, by [2, Eq. (145)]

$$a_l^m = \begin{cases} \sqrt{\dfrac{(l-|m|+1)(l+|m|+1)}{(2l+1)(2l+3)}} & \text{for } l \geq 0 \text{ and } 0 \leq |m| \leq l, \\ 0 & \text{otherwise,} \end{cases} \tag{12}$$

and [2, Eq. (146)]

$$b_l^m = \begin{cases} \sqrt{\dfrac{(l-m-1)(l-m)}{(2l-1)(2l+1)}} & \text{for } l \geq 0 \text{ and } 0 \leq m \leq l, \\ -\sqrt{\dfrac{(l-m-1)(l-m)}{(2l-1)(2l+1)}} & \text{for } l \geq 0 \text{ and } -l \leq m < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

4 Rotation matrices

In this section, we derive a collection of rotation matrices, denoted by $\mathbf{Q}$, which, when applied to a set of ambisonics signals, rotate the sound field around the listener. That is, given a column-vector of ambisonics signals $\mathbf{b}$, the rotated ambisonics signals are given by

$$\mathbf{b}_{\mathrm{r}}(k) = \mathbf{Q} \cdot \mathbf{b}(k). \tag{14}$$

Equivalently, in order to rotate the listener within the sound field, we apply the corresponding inverse rotation matrix. That is, given ambisonics signals $\mathbf{b}$, the desired ambisonics signals for a rotated listener are given by

$$\mathbf{b}_{\mathrm{r}}(k) = \mathbf{Q}^{-1} \cdot \mathbf{b}(k). \tag{15}$$

Note that, unlike translation (as we will see in Section 5), the rotation operation is time- and frequency-independent, so it can be performed either in the frequency domain, where $\mathbf{Q}$ is applied per frequency, or in the time domain, where k then indicates the time sample.

It is convenient to note that all of the rotation matrices presented here are necessarily orthogonal, such that $\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}}$ [4]. Consequently, we can also achieve a rotation of the listener by

$$\mathbf{b}_{\mathrm{r}}(k) = \mathbf{Q}^{\mathrm{T}} \cdot \mathbf{b}(k) \quad \Longleftrightarrow \quad \mathbf{b}_{\mathrm{r}}^{\mathrm{T}}(k) = \mathbf{b}^{\mathrm{T}}(k) \cdot \mathbf{Q}, \tag{16}$$

where $(\cdot)^{\mathrm{T}}$ denotes the transpose of the argument.

We denote the elements of $\mathbf{Q}$ by $Q_{l',l}^{m',m}$ and arrange them as follows:

$$\begin{array}{c} l = \\ l'=0 \\ \\ l'=1 \\ \\ l'=2 \\ \vdots \\ m = \end{array}
\begin{array}{c|ccc|cc}
0 & & 1 & & 2 & \cdots \\
Q_{0,0}^{0,0} & Q_{0,1}^{0,-1} & Q_{0,1}^{0,0} & Q_{0,1}^{0,1} & Q_{0,2}^{0,-2} & \cdots \\
\hline
Q_{1,0}^{-1,0} & Q_{1,1}^{-1,-1} & Q_{1,1}^{-1,0} & Q_{1,1}^{-1,1} & Q_{1,2}^{-1,-2} & \cdots \\
Q_{1,0}^{0,0} & Q_{1,1}^{0,-1} & Q_{1,1}^{0,0} & Q_{1,1}^{0,1} & Q_{1,2}^{0,-2} & \cdots \\
Q_{1,0}^{1,0} & Q_{1,1}^{1,-1} & Q_{1,1}^{1,0} & Q_{1,1}^{1,1} & Q_{1,2}^{1,-2} & \cdots \\
\hline
Q_{2,0}^{-2,0} & Q_{2,1}^{-2,-1} & Q_{2,1}^{-2,0} & Q_{2,1}^{-2,1} & Q_{2,2}^{-2,-2} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
0 & -1 & 0 & 1 & -2 & \cdots
\end{array}, \tag{17}$$

such that the indices l, m correspond to the columns of the matrix while l', m' correspond to the rows. Accordingly, the indices l, m refer to the input ambisonics signals $\mathbf{b}$, while the indices l', m' correspond to the output ambisonics signals $\mathbf{b}_{\mathrm{r}}$.

4.1 Variable yaw rotation

The first rotation matrix we derive is for an arbitrary azimuthal (yaw) rotation around the z-axis. Given a desired rotation angle α, we denote the corresponding rotation matrix by $\mathbf{Q}_{\mathrm{Y}}(\alpha)$ (where the subscript "Y" denotes "yaw"), with elements given by [5, Eq. (3.12)]

$$Q_{l',l}^{m',m}(\alpha) = \begin{cases} \cos m\alpha & \text{if } l = l' \text{ and } m = m', \\ \sin m\alpha & \text{if } l = l' \text{ and } m = -m', \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

The locations of possible nonzero elements of this matrix are illustrated in Fig. 1.

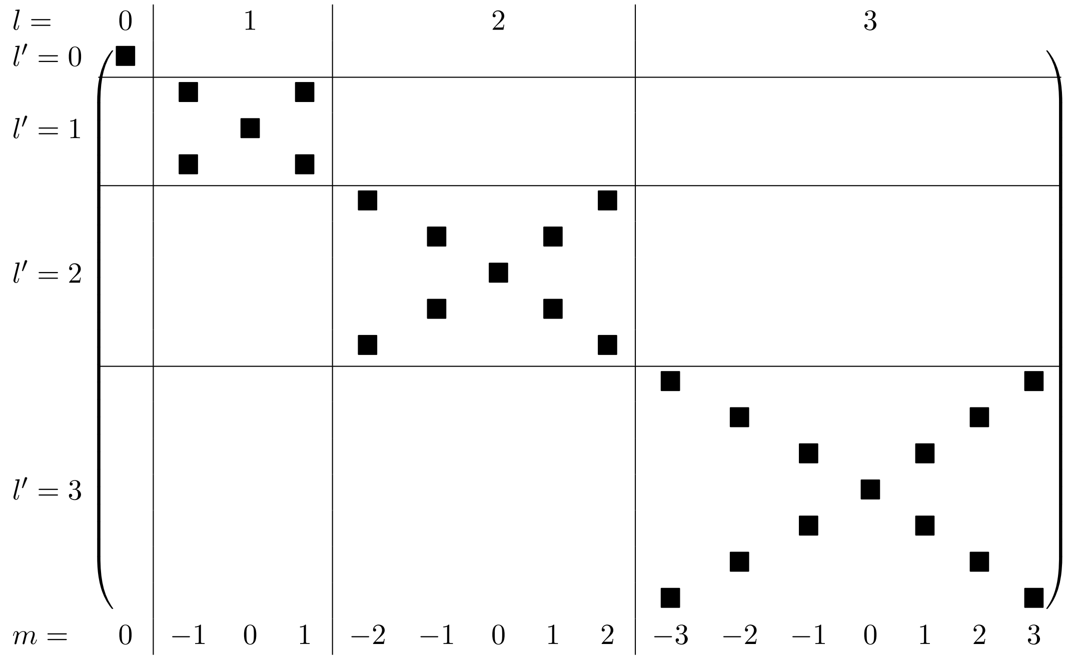

Figure 1: Graphical illustration of the locations of possible nonzero elements (indicated by the ■ symbols) of the rotation matrix for an arbitrary yaw rotation (see Eq. (18)).

4.2 Fixed pitch rotation of $\pi/2$

The second rotation matrix we derive is for a fixed upwards pitch rotation by $\pi/2$ around the y-axis. We denote this matrix $\mathbf{Q}_{\mathrm{P}}(\pi/2)$ (where the subscript "P" denotes "pitch"). As will become clear below, for some of the recurrence formulae used in this derivation, we must compute terms of the matrix beyond the maximum ambisonics order that we intend to use. In particular, to compute $\mathbf{Q}_{\mathrm{P}}(\pi/2)$ up to order L, we must first compute terms in the matrix up to order $2L$, as indicated below.

The procedure for computing the rotation matrix is enumerated below and illustrated symbolically in Fig. 2.

1. First, we find all terms denoted ■, where $m' = 0$, by using [2, Eq. (189)]

$$Q_{l,l}^{0,m} = (-1)^{|m|}\sqrt{2-\delta_{|m|}}\sqrt{\frac{(l-|m|)!}{(l+|m|)!}}P_l^{|m|}(0) \tag{19}$$

and looping over $l \in [0, 2L]$ and $m \in [0, l]$.

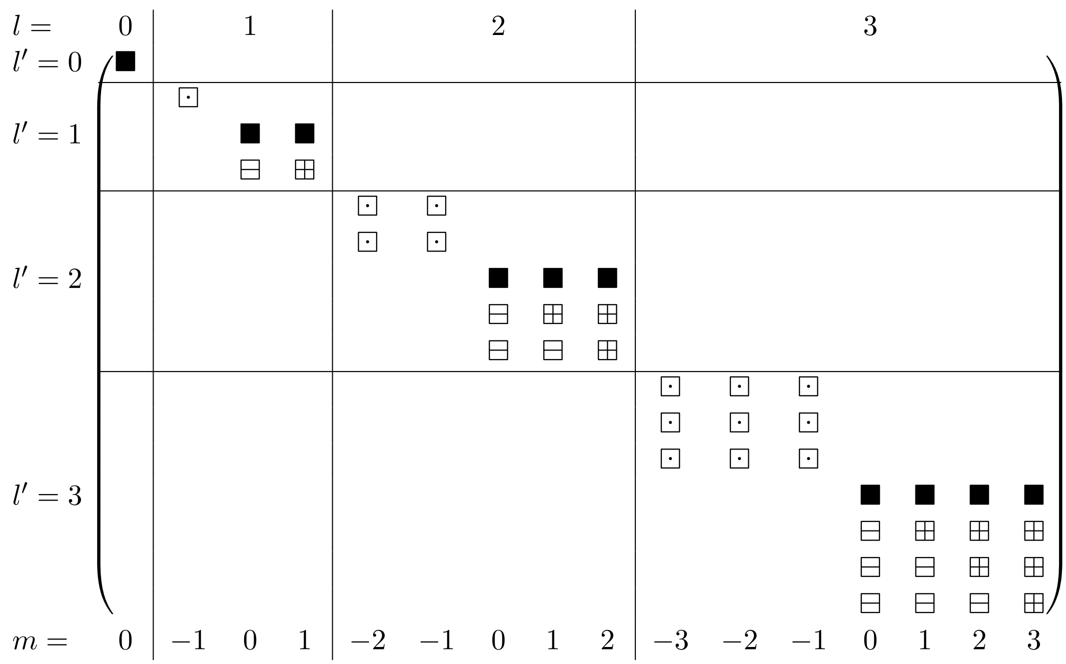

Figure 2: Graphical illustration of the rotation matrix for a pitch rotation of $\pi/2$.

2. Next, we find all terms denoted $\boxplus$ by using [2, Eq. (190)]

$$Q_{l,l}^{m',m} = \frac{\sqrt{2-\delta_{m'}}}{2b_{l+1}^{m'-1}\sqrt{2-\delta_{m'-1}}}\left[\sqrt{2-\delta_m}\left(\frac{b_{l+1}^{m-1}Q_{l+1,l+1}^{m'-1,m-1}}{\sqrt{2-\delta_{m-1}}} - \frac{b_{l+1}^{-m-1}Q_{l+1,l+1}^{m'-1,m+1}}{\sqrt{2-\delta_{m+1}}}\right) + 2a_l^m Q_{l+1,l+1}^{m'-1,m}\right] \quad (20)$$

   and looping over $m' \in [1, L]$, $l \in [m', 2L - m']$, and $m \in [m', l]$.

3. Third, we find all terms denoted $\boxminus$ by using the symmetry relationship [2, Eq. (191)]

$$Q_{l,l}^{m',m} = (-1)^{m'+m} Q_{l,l}^{m,m'} \quad (21)$$

   and looping over $l \in [1, L]$, $m' \in [1, l]$, and $m \in [0, m' - 1]$.

4. Additionally, we find all terms denoted $\boxdot$ by using the symmetry relationship[2]

$$Q_{l,l}^{-m',-m} = Q_{l,l}^{m',m} \quad (22)$$

   and looping over $l \in [1, L]$, $m' \in [1, l]$, and $m \in [1, l]$.

[2]This symmetry relationship in Eq. (22) can be derived by a repeated application of Eqs. (176) and (191) from Zotter [2].

5. Finally, we apply a "mask" to the matrix such that [2, Eq. (187)]

$$Q_{l,l}^{m',m} = \begin{cases} Q_{l,l}^{m',m} & \text{if } m', m \geq 0 \text{ and } (l + m' + m) \text{ is even,} \\ Q_{l,l}^{m',m} & \text{if } m', m < 0 \text{ and } (l + m' + m) \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

4.3 Variable roll rotation

Now we can conveniently compute a matrix for an arbitrary roll rotation of γ around the x-axis. This matrix, which we denote $\mathbf{Q}_\mathrm{R}(\gamma)$ (where the subscript "R" denotes "roll"), is given by

$$\mathbf{Q}_\mathrm{R}(\gamma) = \mathbf{Q}_\mathrm{P}(\pi/2) \cdot \mathbf{Q}_\mathrm{Y}(\gamma) \cdot (\mathbf{Q}_\mathrm{P}(\pi/2))^\mathrm{T}. \tag{24}$$

Recall that the transpose of $\mathbf{Q}_\mathrm{P}(\pi/2)$ (which is exactly equal to the inverse of that matrix) corresponds to a pitch rotation downwards by $\pi/2$.

4.4 Variable pitch rotation

Next, we can conveniently compute a matrix for an arbitrary pitch rotation of β around the y-axis. This matrix, which we denote $\mathbf{Q}_\mathrm{P}(\beta)$, is given by

$$\mathbf{Q}_\mathrm{P}(\beta) = (\mathbf{Q}_\mathrm{R}(\pi/2))^\mathrm{T} \cdot \mathbf{Q}_\mathrm{Y}(\beta) \cdot \mathbf{Q}_\mathrm{R}(\pi/2). \tag{25}$$

4.5 Alignment of the z-axis

Finally, we can compute the rotation matrix needed to orient the z-axis in a specific direction, defined by an azimuth ϕ and elevation θ. This matrix, which we denote $\mathbf{Q}_z(\theta, \phi)$, is given by

$$\mathbf{Q}_z(\theta, \phi) = \mathbf{Q}_\mathrm{Y}(\phi)\mathbf{Q}_\mathrm{P}(\theta - \pi/2). \tag{26}$$

5 Coaxial translation coefficients matrix

As defined in Section 2, we seek a matrix $\mathbf{T}$ of translation coefficients such that, for a given set of ambisonic signals $\mathbf{b}$, the translated ambisonics signals are given by

$$\mathbf{a}(k) = (\mathbf{T}(k, \vec{r}_0))^\mathrm{T} \cdot \mathbf{b}(k). \tag{9}$$

In particular, we seek the matrix of coefficients for a translation of distance d in the $+z$ direction, which we denote $\mathbf{T}_z(k, d\hat{z})$. Thus, the ambisonics signals for a translated listener are given by

$$\mathbf{b}_z(k) = (\mathbf{T}_z(k, d\hat{z}))^\mathrm{T} \cdot \mathbf{b}(k). \tag{27}$$

Similar to the rotation matrix defined above in Section 4, here, we denote the elements of $\mathbf{T}_z$ by $T_{l,l'}^{m,m'}$ and arrange them as follows:

$$
\begin{array}{c}
l' = \\
l = 0 \\
\\
l = 1 \\
\\
l = 2 \\
\vdots \\
m' =
\end{array}
\begin{array}{c|ccc|cc}
0 & & 1 & & 2 & \cdots \\
\hline
T_{0,0}^{0,0} & T_{0,1}^{0,-1} & T_{0,1}^{0,0} & T_{0,1}^{0,1} & T_{0,2}^{0,-2} & \cdots \\
\hline
T_{1,0}^{-1,0} & T_{1,1}^{-1,-1} & T_{1,1}^{-1,0} & T_{1,1}^{-1,1} & T_{1,2}^{-1,-2} & \cdots \\
T_{1,0}^{0,0} & T_{1,1}^{0,-1} & T_{1,1}^{0,0} & T_{1,1}^{0,1} & T_{1,2}^{0,-2} & \cdots \\
T_{1,0}^{1,0} & T_{1,1}^{1,-1} & T_{1,1}^{1,0} & T_{1,1}^{1,1} & T_{1,2}^{1,-2} & \cdots \\
\hline
T_{2,0}^{-2,0} & T_{2,1}^{-2,-1} & T_{2,1}^{-2,0} & T_{2,1}^{-2,1} & T_{2,2}^{-2,-2} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\hline
0 & -1 & 0 & 1 & -2 & \cdots
\end{array}, \tag{28}
$$

such that the indices l, m correspond to the rows of the matrix while l', m' correspond to the columns. Although this indexing convention appears reversed compared to that for the rotation matrix (cf. Eq. (17)), due to the matrix transpose in Eq. (27), we again have that the indices l, m refer to the input ambisonics signals $\mathbf{b}$, while the indices l', m' correspond to the output ambisonics signals $\mathbf{b}_z$.

The procedure for computing the matrix of coaxial translation coefficients is enumerated below[3] and illustrated symbolically in Fig. 3. Note that due to the symmetry granted by translating along the z-axis, we have $T_{l,l'}^{m,m'} = 0$ for all $m' \neq m$. Consequently, in describing the algorithm for computing this matrix, we consider only those terms where $m' = m$.

1. We first find the terms denoted ■ by using [2, Eq. (166)][4]

$$T_{0,l'}^{0,0}(k; d\hat{z}) = (-1)^{l'} \sqrt{2l'+1}\, j_{l'}(kd) \tag{29}$$

   and looping over $l' \in [0, 2L]$.

2. Next, we find the terms denoted ⊠ by using [2, Eq. (163b)]

$$b_l^{-m} T_{l,l'}^{m,m} = -b_{l'+1}^{m-1} T_{l-1,l'+1}^{m-1,m-1} + b_{l'}^{-m} T_{l-1,l'-1}^{m-1,m-1} + b_{l-1}^{m-1} T_{l-2,l'}^{m,m} \tag{30}$$

   and looping over $l \in [1, L]$ and $l' \in [l, 2L - l]$ with $l = m$. Note, however, that since $l = m$, the above expression reduces to

$$b_l^{-m} T_{l,l'}^{m,m} = -b_{l'+1}^{m-1} T_{l-1,l'+1}^{m-1,m-1} + b_{l'}^{-m} T_{l-1,l'-1}^{m-1,m-1} + b_{l-1}^{m-1} \cancel{T_{l-2,l'}^{m,m}} \tag{31}$$

   since no such coefficients $T_{l,l'}^{m,m'}$ exist where $m > l$.

[3]Throughout this section, we refer to the equations given by Zotter [2] for complex-valued spherical harmonics, which are identical to those for the real-valued spherical harmonics. The equations corresponding specifically to real-valued spherical harmonics can be found in the unnumbered set of equations following Eq. (185) on p. 53 of Zotter [2]. Also note that, compared to Zotter [2], the order of the indices for our translation coefficients $T_{l,l'}^{m,m'}$ is reversed, such that our indices correspond to (row, column) rather than (column, row). Our use of the "primed" indices (e.g., l'), however, is the same as in Zotter [2].

[4]Note that Zotter [2] erroneously omits the "prime" in the order of the spherical Bessel function in Eq. (166).

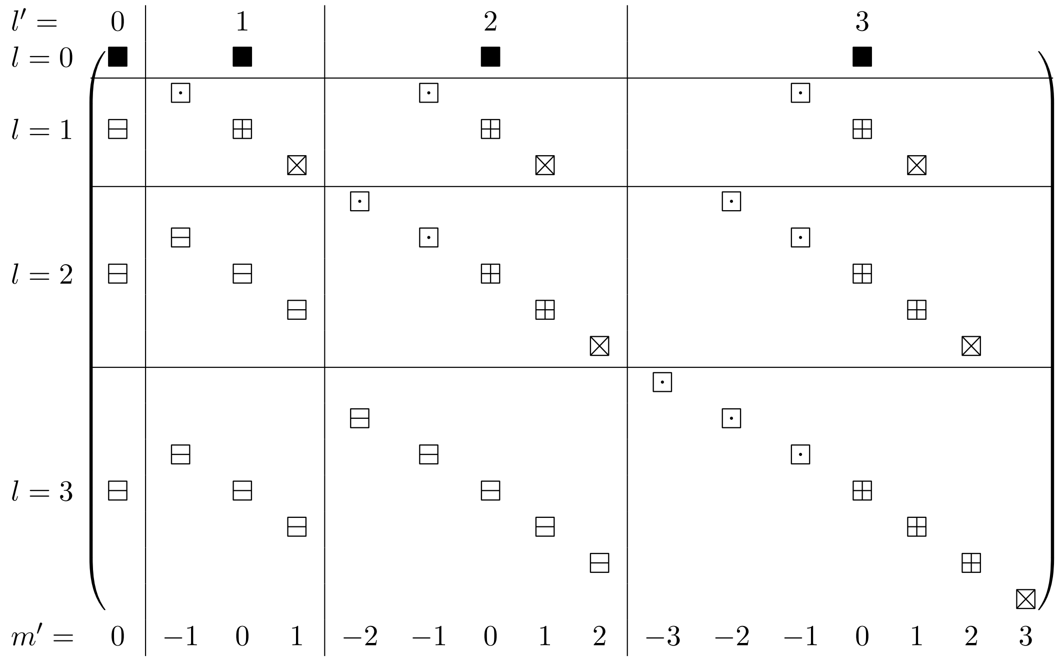

Figure 3: Graphical illustration of the translation coefficient matrix for translating along the z-axis.

3. Third, we find the terms denoted $\boxplus$ by using [2, Eq. (163a)]

$$a_{l-1}^{m} T_{l,l'}^{m,m} = -a_{l'}^{m} T_{l-1,l'+1}^{m,m} + a_{l'-1}^{m} T_{l-1,l'-1}^{m,m} + a_{l-2}^{m} T_{l-2,l'}^{m,m} \tag{32}$$

and looping over $m \in [0, L-1]$, $l \in [m+1, L]$, and $l' \in [l, 2L-l]$. Note that the third term on the right hand side will go to zero when $l-2 < m$ (see Eq. (12)).

4. We then find the terms denoted $\boxdot$ by using [2, Eq. (161)]

$$T_{l,l'}^{-m,-m} = T_{l,l'}^{m,m} = T_{l,l'}^{|m|,|m|} \tag{33}$$

and looping over $l \in [1, L]$, $l' \in [l, L]$, and $m \in [1, l]$.

5. Fifth, we find the terms denoted $\boxminus$ by using [2, Eq. (162)]

$$T_{l,l'}^{m,m} = (-1)^{l+l'} T_{l',l}^{m,m} \tag{34}$$

and looping over $l' \in [0, L-1]$, $l \in [l'+1, L]$, and $m \in [-l', l']$.

6. Finally, due to our choice to include a factor of $(-i)^l$ in our spherical Fourier-Bessel series basis functions (see Eq. (1)), we multiply every nonzero term in the matrix $\mathbf{T}$ by $(-i)^{l-l'}$.

6 General translation coefficients matrix

Combining the results of the previous sections, we can now write the translation coefficients matrix for an arbitrary translation of distance r_0 in the direction (θ, ϕ). Using Eqs. (16) and (27), the combined translation coefficient matrix is then given by

$$\begin{aligned} \left(\mathbf{T}(k, \vec{r}_0)\right)^{\mathrm{T}} &= \mathbf{Q}_z(\theta, \phi) \cdot \left(\mathbf{T}_z(k, r_0 \hat{z})\right)^{\mathrm{T}} \cdot \left(\mathbf{Q}_z(\theta, \phi)\right)^{\mathrm{T}}, \\ \implies \mathbf{T}(k, \vec{r}_0) &= \mathbf{Q}_z(\theta, \phi) \cdot \mathbf{T}_z(k, r_0 \hat{z}) \cdot \left(\mathbf{Q}_z(\theta, \phi)\right)^{\mathrm{T}} \end{aligned} \tag{35}$$

where $\mathbf{Q}_z$ is given by Eq. (26) and $\mathbf{T}_z$ is found as described in Section 5.

References

[1] Nail A. Gumerov and Ramani Duraiswami. *Fast Multipole Methods for the Helmholtz Equation in Three Dimensions.* Elsevier Science, 2005.

[2] Franz Zotter. *Analysis and Synthesis of Sound-Radiation with Spherical Arrays.* PhD thesis, University of Music and Performing Arts Graz, 2009.

[3] Christian Nachbar, Franz Zotter, Etienne Deleflie, and Alois Sontacchi. ambiX - A Suggested Ambisonics Format. In *Proceedings of the 3rd Ambisonics Symposium*, 6 2011. URL `http://ambisonics.iem.at/proceedings-of-the-ambisonics-symposium-2011/ambix-a-suggested-ambisonics-format`.

[4] Cheol Ho Choi, Joseph Ivanic, Mark S. Gordon, and Klaus Ruedenberg. Rapid and stable determination of rotation matrices between spherical harmonics by direct recursion. *The Journal of Chemical Physics*, 111(19):8825–8831, 1999. doi: 10.1063/1.480229. URL `https://doi.org/10.1063/1.480229`.

[5] Matthias Kronlachner. Spatial Transformations for the Alteration of Ambisonic Recordings. Master's thesis, Graz University of Technology, 2014.